Brownian motion and Stochastic Calculus
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Assignment 2—solutions

Exercise 1

Let $(\Omega,\mathcal{F},\mathbb{P})$ be a probability space and let X, Y, and Z be random variables and suppose that Z is $\sigma(X,Y)$-measurable. Use the monotone class theorem to show that there exists a measurable function $f:\mathbb{R}^2\longrightarrow\mathbb{R}$ such that $Z=f(X,Y)$.

Let us first assume that Z is bounded. Define

- **$\mathcal{H}$ as the set of bounded random variables of the form $\tilde{Z}=f(X,Y)$ for some measurable bounded function $f:\mathbb{R}^2\to\mathbb{R}$,**
- $\mathcal{A}=\{\mathbf{1}_{\{X\in A\}}\mathbf{1}_{\{Y\in B\}}:A,B\in\mathcal{B}(\mathbb{R})\}$.

It is clear that $\mathcal{A}$ is stable by multiplication and that $\mathcal{H}$ is a vector space containing constant function 1 as well as the set $\mathcal{A}$. We verify that $\mathcal{H}^+=\{\tilde{Z}\in\mathcal{H}:\tilde{Z}\geq 0\}$ is stable by taking bounded non-decreasing limits.

Let us have a sequence $(Z_k)\subset\mathcal{H}^+$ and such that $0\leq Z_1\leq\ldots\leq Z_n\leq\ldots\leq C$ for some finite constant $C>0$. Since $(Z_k)\subset\mathcal{H}^+\subset\mathcal{H}$, there exist measurable bounded functions f_k, $k\in\mathbb{N}$, such that $Z_k=f_k(X,Y)$, $k\in\mathbb{N}$. Let us denote $\tilde{Z}:=\lim_{k\to\infty}Z_k$. By monotonicity, $\tilde{Z}=\sup_{k\in\mathbb{N}}Z_k$. It then suffices to take $f(x,y):=\sup_{k\in\mathbb{N}}f_k(x,y)\vee C$ to get $\tilde{Z}=f(X,Y)$. Hence, $\tilde{Z}\in\mathcal{H}^+$ and therefore $\mathcal{H}^+$ is stable by taking bounded non-decreasing limits. Because $\sigma(\mathcal{A})=\sigma(X,Y)$, monotone class theorem then yields that $\mathcal{H}$ contains all bounded $\sigma(X,Y)$-measurable functions.

Let now Z be a general $\sigma(X,Y)$-measurable random variable. Clearly, $Z^{n+}:=Z^+\mathbf{1}_{\{|Z^+|\leq n\}}$ and $Z^{n-}:=Z^-\mathbf{1}_{\{|Z^-|\leq n\}}$ are bounded and $\sigma(X,Y)$-measurable for every $n\in\mathbb{N}$. By the first part, there are measurable functions f^{n+} and f^{n-} such that $Z^{n+}=f^{n+}(X,Y)$ and $Z^{n-}=f^{n-}(X,Y)$.

It follows that

$$Z=Z^+-Z^-=\sup_{n\in\mathbb{N}}Z^{n+}-\sup_{n\in\mathbb{N}}Z^{n-}=\sup_{n\in\mathbb{N}}f^{n+}(X,Y)-\sup_{n\in\mathbb{N}}f^{n-}(X,Y).$$

It is then clear that we can take

$$f(x,y)=\left(\sup_{n\in\mathbb{N}}f^{n+}(x,y)\right)\mathbf{1}_{\left\{\left|\sup_{n\in\mathbb{N}}f^{n+}(x,y)\right|<\infty\right\}}-\left(\sup_{n\in\mathbb{N}}f^{n-}(x,y)\right)\mathbf{1}_{\left\{\left|\sup_{n\in\mathbb{N}}f^{n-}(x,y)\right|<\infty\right\}}$$

to get $Z=f(X,Y)$.

Exercise 2

Fix two measurable processes X and Y on some probability space $(\Omega,\mathcal{F},\mathbb{P})$.

1) Assume that X and Y are both right-continuous or both left-continuous. Show that they are $\mathbb{P}$-modifications of each other if and only if they are $\mathbb{P}$-indistinguishable.

2) Show that the previous result is not true in general.

1) **We just show that the fact that X is a version of Y implies the indistinguishability, since the converse is obvious. Without loss of generality, we assume that X and Y are right-continuous.**

For $t \geq 0$, we define the null set $N_t := \{\omega : X_t(\omega) \neq Y_t(\omega)\}$. We consider $N := \cup_{t\in\mathbb{Q}_+} N_t$, which remains a null set as a countable union of null sets. Finally, we introduce the null set $A_Z := \{\omega : Z_\cdot(\omega)$ not right-continuous$\}$ for $Z = X, Y$ and we define $M := A_X \cup A_Y \cup N$, which is still a null set.

It suffices to check that, for all $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\ t \geq 0$. By definition of M we clearly have that, for $\omega \in M^c$, $X_t(\omega) = Y_t(\omega)\ \forall\ t \in \mathbb{Q}_+$. Now, take any $t \geq 0$ and let (t_n) be a sequence in $\mathbb{Q}_+$ with $t_n \downarrow t$. The right-continuity of the paths $X_\cdot(\omega)$ and $Y_\cdot(\omega)$ then implies $X_t(\omega) = \lim_{n\to\infty} X_{t_n}(\omega) = \lim_{n\to\infty} Y_{t_n}(\omega) = Y_t(\omega)$.

2) **Take $\Omega = [0,\infty)$, $\mathcal{F} = \mathcal{B}([0,\infty))$ the Borel σ-algebra, and P a probability measure with $P(\{\omega\}) = 0,\ \forall \omega \in \Omega$ (for instance, the exponential distribution). Set $X \equiv 0$ and**

$$Y_t(\omega) = \begin{cases} 1, & t = \omega, \\ 0, & \textbf{else.} \end{cases}$$

Then, $\mathbb{P}[X_t = Y_t] = 1$, $\forall t \geq 0$, since single points have no mass, but $\{X_t = Y_t,\ \forall t \geq 0\} = \emptyset$. Note that all sample paths of X are continuous, while all sample paths of Y are discontinuous at $t = \omega$.

Exercise 3

Let X be a process on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ satisfies the usual conditions. We want to show

$$X\ \mathbb{F}\text{-optional} \Longrightarrow X\ \mathbb{F}\text{–progressively measurable} \Longrightarrow X\ \mathbb{F}\text{-adapted and measurable.}$$

1) Show that every $\mathbb{F}$–progressively measurable process is $\mathbb{F}$-adapted and measurable.

2) Assume that X is $\mathbb{F}$-adapted and that every path of X is right-continuous (resp. left-continuous). Show that X is $\mathbb{F}$–progressively measurable.

3) Show that $\mathcal{O}(\mathbb{F})$ is generated by all bounded, càdlàg, $\mathbb{F}$-adapted and measurable processes.

4) Use the monotone class theorem to show that every $\mathbb{F}$-optional process is $\mathbb{F}$–progressively measurable.

1) **Let X be $\mathbb{F}$–progressively measurable. Then $X\mathbf{1}_{\Omega\times[0,t]}$ is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable for every $t \geq 0$. For any $t \geq 0$, we see that $X_t = X \circ i_t$, where $i_t : (\Omega, \mathcal{F}_t) \longrightarrow (\Omega\times[0,t], \mathcal{F}_t\otimes\mathcal{B}[0,t])$, $\omega \longmapsto (\omega, t)$ is measurable. Therefore, X_t is $\mathcal{F}_t$-measurable for every $t \geq 0$. Moreover, the processes X^n defined by $X^n_u := X\mathbf{1}_{\Omega\times[0,n]}\mathbf{1}_{[0,n]}(u)$, $u \geq 0$, are $\mathcal{F}\otimes\mathcal{B}[0,\infty)$-measurable. Since $X^n \to X$ pointwise (in (t,ω)) as $n \to \infty$, also X is $\mathcal{F}\otimes\mathcal{B}[0,\infty)$-measurable.**

2) **Fix a $t \geq 0$ and consider the sequence of processes Y^n on $\Omega \times [0,t]$ given by $Y^n_0 = X_0$ and**

$$Y^n_u := \sum_{k=1}^{2^n-1} 1_{(tk2^{-n}, t(k+1)2^{-n}]}(u) X_{t(k+1)2^{-n}}, \textbf{ for } u \in (0,t].$$

By construction, each Y^n is $\mathcal{F}_t \otimes \mathcal{B}[0,t]$-measurable. Since $Y^n \to X|_{\Omega\times[0,t]}$ pointwise as $n \to \infty$ due to right-continuity, the result follows.

3) **Let X be adapted, with all paths being càdlàg. Consider the processes $X^n := (X \wedge n) \vee (-n)$. Clearly, each X^n is bounded and càdlàg. Thus, each X^n is $\sigma(\mathcal{M})$-measurable. As the pointwise limit of the X^n, also X is $\sigma(\mathcal{M})$-measurable. It follows that $\mathcal{O} \subset \sigma(\mathcal{M})$. The reverse inclusion is trivial.**

4) **If a process X is optional, then $X^n := X\,1_{\{|X|\leq n\}}$ is also optional and of course $X^n \to X$; so if each X^n is progressively measurable, then so is X, and hence we can assume without loss of generality that X is bounded. Let $\mathcal{H}$ denote the real vector space of bounded, progressively measurable processes. By 2), $\mathcal{H}$ contains $\mathcal{M}$. Clearly, $\mathcal{H}$ contains the constant process 1 and is closed under monotone bounded convergence. Also, $\mathcal{M}$ is closed under multiplication. The monotone class theorem yields that every**

bounded $\sigma(\mathcal{M})$-measurable process is contained in $\mathcal{H}$. Due to 3)**, we conclude that every bounded optional process is progressively measurable.**

Exercise 4

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and B a $\mathbb{P}$–Brownian motion on $[0,1]$. Let $k \in \mathbb{N}^\star$, and $0 = s_1 < t_1 < s_2 < t_2 < \cdots < t_k < s_{k+1} = 1$. Find the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ conditional on $(B_{s_1}, \ldots, B_{s_{k+1}})$.

Let $D := \{a2^{-m} : m \in \mathbb{N},\ a \in \{0,1,\ldots,2^m\}\}$. Let Z_1, Z_2,... be an infinite sequence of i.i.d. standard normal random variables. Construct in terms of the Z_j a stochastic process $(W_t)_{t\in D}$ such that the law of W is equal to the law of $(B_t)_{t\in D}$.

1) **Note that** $(B_{s_1}, B_{t_1}, \ldots, B_{t_k}, B_{s_{k+1}})$ **is a Gaussian vector. We now claim that for each** $k \in \mathbb{N}^\star$**, the random variable**

$$\Delta_k := B_{t_k} - \frac{t_k - s_k}{s_{k+1} - s_k} B_{s_{k+1}} - \frac{s_{k+1} - t_k}{s_{k+1} - s_k} B_{s_k},$$

is normally distributed with mean 0 **and variance** $(s_{k+1} - t_k)(t_k - s_k)/(s_{k+1} - s_k)$**. In addition,** Δ_k **is** $\mathbb{P}$**-independent of** $(B_{s_1}, \ldots, B_{s_{k+1}})$**.**

Indeed, the first claim is direct form the Gaussian vector property, as well as the equality

$$\Delta_k = -\frac{t_k - s_k}{s_{k+1} - s_k}\big(B_{s_{k+1}} - B_{t_k}\big) - \frac{s_{k+1} - t_k}{s_{k+1} - s_k} B_{s_k},$$

which allows to easily compute the variance. As for the second claim, it is enough to show that Δ_k **is uncorrelated with** $B_{s_{j+1}} - B_{s_j}$**, for any** $j \in \{1, \ldots, k\}$**, which is direct by computations.**

We conclude that the law of $(B_{t_1}, B_{t_2}, \ldots, B_{t_k})$ **conditional on** $(B_{s_1}, \ldots, B_{s_{k+1}})$ **is Gaussian with mean vector** μ **with**

$$\mu^k := \frac{t_k - s_k}{s_{k+1} - s_k} B_{s_{k+1}} + \frac{s_{k+1} - t_k}{s_{k+1} - s_k} B_{s_k},\ k \in \mathbb{N}^\star,$$

and variance–covariance matrix Σ **which is diagonal with**

$$\Sigma^{k,k} := \frac{(s_{k+1} - t_k)(t_k - s_k)}{s_{k+1} - s_k}.$$

2) **Let** $\mathcal{D}^n := \{a2^{-m} : m \in \{1, \ldots, n\},\ a \in \{0, 1, \ldots, 2z^m\}\}$**. We construct W recursively on each** $\mathcal{D}^n$**, so that finally we obtain** W **on** $\mathcal{D}$**. The first step is to define** $W_1 := Z_1$**, so that clearly** $W \overset{\text{law}}{=} B$ **on** $\{0,1\}$**. If we have defined** W **on** $\mathcal{D}^n$ **in terms of** $(Z_1, Z_2, \ldots, Z_{2^{n-1}})$**, we extend it to** $\mathcal{D}^{n+1}$ **by**

$$W_{(2j-1)2^{-(m+1)}} := \frac{1}{2} W_{(j-1)2^{-m}} + \frac{1}{2} W_{j2^{-m}} + 2^{-n/2-1} Z_{2^n+j},\ j \in \{1, \ldots, 2^n\}.$$

By induction, assume that $W \overset{\text{law}}{=} B$ **on** $\mathcal{D}^n$**. We also obtain from this construction that the law of** $W|_{\mathcal{D}^{n+1}}$ **conditional on** $W|_{\mathcal{D}^n}$ **is equal to the law of** $B|_{\mathcal{D}^{n+1}}$ **conditional on** $B|_{\mathcal{D}^n}$**, by** 1)**. Therefore, the inductive step is valid, and we finally obtain that the law of** W **is equal to the law of** $B|_{\mathcal{D}}$ **by the Ionescu–Tulcea theorem.**